ON METRIC, DYNAMICAL, PROBABILISTIC AND FRACTAL PHENOMENA CONNECTED WITH THE SECOND OSTROGRADSKY EXPANSION

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Abstract. We consider the second Ostrogradsky expansion from the number theory, probability theory, dynamical systems and fractal geometry points of view, and establish several new phenomena connected with this expansion.

First of all we prove the singularity of the random second Ostrogradsky expansion.

Secondly we study properties of the symbolic dynamical system generated by the natural one-sided shift-transformation T on the second Ostrogradsky expansion. It is shown, in particular, that there are no probability measures which are simultaneously invariant and ergodic (w.r.t. T) and absolutely continuous (w.r.t. Lebesgue measure). So, the classical ergodic approach to the development of the metric theory is not applicable for the second Ostrogradsky expansion.

We develop instead the metric and dimensional theories for this expansion using probabilistic methods. We show, in particular, that for Lebesgue almost all real numbers any digit i from the alphabet $A = \mathbb{N}$ appears only finitely many times in the difference-version of the second Ostrogradsky expansion, and the set of all reals with bounded digits of this expansion is of zero Hausdorff dimension.

Finally, we compare metric, probabilistic and dimensional theories for the second Ostrogradsky expansions with the corresponding theories for the Ostrogradsky-Pierce expansion and for the continued fractions expansion.

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AMS Subject Classifications (2010): 11K55, 28A80, 37A45, 37B10, 60G30.

Key words: the second Ostrogradsky expansion, symbolic dynamical systems, singular probability measures, Hausdorff dimension, Ostrogradsky-Pierce expansion, continued fractions.

1. Introduction

The presented paper is devoted to the investigation of the expansions of real numbers in the Ostrogradsky series (they were introduced by M. V. Ostrogradsky, a well known ukrainian mathematician who lived from 1801 to 1862).

The expansion of x of the form:

$$x = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{n-1}}{q_1 q_2 \cdots q_n} + \cdots , \tag{1}$$

where q_n are positive integers and $q_{n+1} > q_n$ for all n, is said to be the expansion of x in the first Ostrogradsky series. The expansion of x of the form:

$$x = \frac{1}{q_1} - \frac{1}{q_2} + \cdots + \frac{(-1)^{n-1}}{q_n} + \cdots , \tag{2}$$

where q_n are positive integers and $q_{n+1} \geq q_n(q_n + 1)$ for all n, is said to be the expansion of x in the second Ostrogradsky series. Each irrational number has a unique expansion of the form (1) or (2). Rational numbers have two finite different representations of the above form (see, e.g., [25]).

Shortly before his death, M. Ostrogradsky has proposed two algorithms for the representation of real numbers via alternating series of the form (1) and (2), but he did not publish any papers on this problems. Short Ostrogradsky's remarks concerning the above representations have been found by E. Ya. Remez [25] in the hand-written fund of the Academy of Sciences of USSR. E. Ya. Remez has pointed out some similarities between the Ostrogradsky series and continued fractions. He also paid a great attention to the applications of the Ostrogradsky series for the numerical methods for solving algebraic equations. In the editorial comments to the book [22] B. Gnedenko has pointed out that there are no fundamental investigations of properties of the above mentioned representations. Since that time there were a lot of publications devoted to the first Ostrogrdsky series (see, e.g., [1, 11] and references and historical notes therein). Unfortunately the metric theory of the second Ostrogradsky is still not well developed.

The second Ostrogradsky series converges rather quickly, giving a very good approximation of irrational numbers by rationals, which are partial sums of the above series.

The representation of a real number x in the form (2) is said to be the O^2-*expansion* of x and will be denoted by $O^2(q_1(x), \ldots, q_n(x), ...)$.

For a given set of k initial symbols, the (k+1)-th symbol of the O^2-expansion can not take the values $1, 2, 3, 4, 5, ..., q_k(q_k+1)-1$, which makes these O^2-symbols dependent.

Let

$$d_1 = q_1 \quad \text{and} \quad d_{k+1} = q_{k+1} - q_k(q_k+1) + 1 \quad \forall k \in N.$$

Then the previous series can be rewritten in the form

$$x = \sum_k \frac{(-1)^{k+1}}{q_{k-1}(x)(q_{k-1}(x)+1) - 1 + d_k(x)} =: \bar{O}^2(d_1(x), d_2(x), \dots, d_k(x), \dots). \tag{3}$$

Expression (3) is said to be the $\bar{O}^2$-expansion (the second Ostrogradsky expansion with independent increments). The number $d_k = d_k(x)$ is said to be the k-th $\bar{O}^2$-symbol of x. In the $\bar{O}^2$-expansion any symbol (independently of all previous ones) can be chosen from the whole set of positive integers.

The main aims of the present paper are:

1) to develop ergodic, metric and dimensional theories of the $\bar{O}^2$-expansion for real numbers and compare these theories with the corresponding theories for the continued fraction expansion and for the Ostrogradsky-Pierce expansion;

2) to study properties of the symbolic dynamical system generated by the one-sided shift transformation on the $\bar{O}^2$-expansion:

$$\forall\ x = \bar{O}^2(d_1(x), d_2(x), \dots, d_n(x), \dots) \quad \in [0,1],$$

$$T(x) = T(\bar{O}^2(d_1(x), d_2(x), \dots, d_n(x), \dots)) = \bar{O}^2(d_2(x), d_3(x), \dots, d_n(x), \dots);$$

3) to study the distributions of random variables

$$\eta = \bar{O}^2(\eta_1, \eta_2, ..., \eta_n, ...),$$

whose $\bar{O}^2$-symbols η_k are independent identically distributed random variables taking the values $1, 2, \dots, m, \dots$ with probabilities $p_1, p_2, \dots, p_m, \dots$ respectively, $p_m \geq 0$, $\sum\limits_{m=1}^{\infty} p_m = 1$.

2. Properties of cylinders of the O^2- and the $\bar{O}^2$-expansions

Let $(c_1, c_2, \dots, c_n)$ be a given set of natural numbers. The set

$$\Delta^{O^2}_{c_1 c_2 \dots c_n} = \{x : x = O^2(q_1, q_2, \dots, q_n, \dots), q_i = c_i, i = \overline{1,n}, q_{n+j} \in N\}$$

is said to be the cylinder of rank n with the base $c_1 c_2 \dots c_n$

Let us mention some properties of the O^2-cylinders.

1. $\Delta^{O^2}_{c_1 \dots c_n} = \bigcup\limits_{i=c_n(c_n+1)}^{\infty} \Delta^{O^2}_{c_1 \dots c_n i}.$

2. $\inf \Delta^{O^2}_{c_1...2m-1} = \sum\limits_{k=1}^{2m-1} \frac{(-1)^{k-1}}{c_k} - \frac{1}{c_{2m-1}(c_{2m-1}+1)} = O^2(c_1,\dots,c_{2m-1},c_{2m-1}(c_{2m-1}+1)) = O^2(c_1,\dots,c_{2m-2},c_{2m-1}+1) \in \Delta^{O^2}_{c_1\dots c_{2m-1}}$;
   $\sup \Delta^{O^2}_{c_1...2m-1} = \sum\limits_{k=1}^{2m-1} \frac{(-1)^{k-1}}{c_k} = O^2(c_1,\dots,c_{2m-1}) \in \Delta^{O^2}_{c_1\dots c_{2m-1}}$;
   $\inf \Delta^{O^2}_{c_1...2m} = \sum\limits_{k=1}^{2m} \frac{(-1)^{k-1}}{c_k} = O^2(c_1,\dots,c_{2m}) \in \Delta^{O^2}_{c_1\dots c_{2m}}$;
   $\sup \Delta^{O^2}_{c_1...2m} = \sum\limits_{k=1}^{2m} \frac{(-1)^{k-1}}{c_k} + \frac{1}{c_{2m}(c_{2m}+1)} = O^2(c_1,\dots,c_{2m},c_{2m}(c_{2m}+1)) = O^2(c_1,\dots,c_{2m-1},c_{2m}+1) \in \Delta^{O^2}_{c_1\dots c_{2m}}$.
3. $\sup \Delta^{O^2}_{c_1\dots c_{2m-1}i} = \inf \Delta^{O^2}_{c_1\dots c_{2m-1}(i+1)}$;
   $\inf \Delta^{O^2}_{c_1\dots c_{2m}i} = \sup \Delta^{O^2}_{c_1\dots c_{2m}(i+1)}$.

Lemma 1. *The cylinder $\Delta^{O_2}_{c_1\dots c_n}$ is a closed interval $[a,b]$, where*

$$a = \begin{cases} O_2(c_1,\dots,c_n,c_n(c_n+1)), & \text{if } n \text{ is odd} \quad ,\\ O_2(c_1,\dots,c_n), & \text{if } n \text{ is even} \quad ;\end{cases}$$

$$b = \begin{cases} O_2(c_1,\dots,c_n), & \text{if } n \text{ is odd} \quad ,\\ O_2(c_1,\dots,c_n(c_n+1)), & \text{if } n \text{ is even} \quad .\end{cases}$$

Proof. It is clear that $\Delta^{O_2}_{c_1\dots c_n} \subset [a,b]$. Let us prove that $[a,b] \subset \Delta^{O_2}_{c_1\dots c_n}$. Since $a,b \in \Delta^{O_2}_{c_1\dots c_n}$, it is enough to show that any $x \in (a,b)$belongs to $\Delta^{O_2}_{c_1\dots c_n}$.

If $a < x < b$, then

$$x = a + x_1, \text{ where } a < x_1 < b - a = \frac{1}{c_n(c_n+1)}.$$

The Ostrogradsky-Remez theorem states that x_1 can be decomposed into the second Ostrogradsky series by using the following algorithm:

$$\begin{cases} 1 = q_1 x + \beta_1 \quad (0 \le \beta_1 < x),\\ q_1 = q_2\beta_1 + \beta_2 \quad (0 \le \beta_2 < \beta_1),\\ q_2 q_1 = q_3\beta_2 + \beta_3 \quad (0 \le \beta_3 < \beta_2),\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ q_k \dots q_2 q_1 = q_{k+1}\beta_k + \beta_{k+1} \quad (0 \le \beta_{k+1} < \beta_k),\\ \dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

i.e.,

$$x_1 = \frac{1}{q_1'} - \frac{1}{q_2'} + \frac{1}{q_3'} - \dots + (-1)^{k-1}\frac{1}{q_k'} + \dots .$$

Let us show that $q_1' \ge c_n(c_n+1)$. Since

$$\frac{1}{q_1'+1} \le \frac{1}{q_1'} - \frac{1}{q_2'} \le x_1 \le \frac{1}{q_1'},$$

we get
$$\frac{1}{q_1'+1} \leq x_1 < \frac{1}{c_n(c_n+1)},$$
and hence, $c_n(c_n+1) < q_1'+1$ and $c_n(c_n+1) \leq q_1'$.

So, the number x has the following O_2-expansion: $x = O_2(c_1, \ldots, c_n, q_1', q_2', \ldots)$, and, therefore, $x \in \Delta^{O_2}_{c_1\ldots c_n}$, which proves the lemma. □

Corollary 1. *For the length of the cylinder of rank n the following relations hold:*
$$|\Delta^{O^2}_{c_1\ldots\ldots c_n}| = \frac{1}{c_n(c_n+1)} \to 0 \quad (n \to \infty);$$

Corollary 2. *The length of a cylinder depends only on the last digit of the base:*
$$|\Delta^{O^2}_{c_1\ldots c_n i}| = \frac{1}{i(i+1)} = |\Delta^{O^2}_{s_1\ldots s_m i}|, \quad \forall i \geq \max\{c_n(c_n+1), s_m(s_m+1)\}.$$

Any cylinder of the O^2-expansion can be rewritten in terms of the $\bar{O}^2$-expansion:
$$\Delta^{O^2}_{c_1\ldots c_n} \equiv \bar{\Delta}^{O^2}_{a_1\ldots\ldots a_n},$$
where $a_1 = c_1,\ a_k = c_k + 1 - c_{k-1}(c_{k-1}+1),\ 1 < k < n$.

Properties of O^2-cylinders generate properties of the $\bar{O}^2$-cylinders. Let us mention only some of these properties.

Lemma 2.
$$\frac{|\Delta^{\bar{O}^2}_{a_1\ldots a_n i}|}{|\Delta^{\bar{O}^2}_{a_1\ldots a_n}|} = \frac{|\Delta^{O^2}_{c_1\ldots c_n(c_n(c_n+1)-1+i)}|}{|\Delta^{O^2}_{c_1\ldots c_n}|} \leq \frac{1}{c_n(c_n+1)+i} < \frac{1}{c_n^2}.$$

Proof.
$$\frac{|\Delta^{\bar{O}^2}_{a_1\ldots a_n i}|}{|\Delta^{\bar{O}^2}_{a_1\ldots a_n}|} = \frac{|\Delta^{O^2}_{c_1\ldots c_n(c_n(c_n+1)-1+i)}|}{|\Delta^{O^2}_{c_1\ldots c_n}|} = \frac{c_n(c_n+1)}{(c_n(c_n+1)+i-1)(c_n(c_n+1)+i)} =$$
$$= \frac{1}{\left(1+\frac{i-1}{c_n(c_n+1)}\right)(c_n(c_n+1)+i)} \leq \frac{1}{c_n(c_n+1)+i} < \frac{1}{c_n^2}.$$
□

Remark 1. For the continued fractions
$$\frac{1}{3i^2} \leq \frac{|\Delta^{c.f.}_{a_1\ldots a_n i}|}{|\Delta^{c.f.}_{a_1\ldots a_n}|} \leq \frac{1}{i^2},$$
independently of the previous digits.

Lemma 3. *For any sequence of positive integers $a_1, a_2, ..., a_k$ the following inequalities hold:*
$$\frac{|\Delta^{\bar{O}^2}_{a_1a_2\ldots a_k j}|}{|\Delta^{\bar{O}^2}_{a_1a_2\ldots a_k}|} < \frac{1}{2^{2^{k-1}}},\ \forall j \in N;$$

$$\frac{|\Delta^{\bar{O}^2}_{ba_2\ldots a_k j}|}{|\Delta^{\bar{O}^2}_{ba_2\ldots a_k}|} < \frac{1}{b^{2^k}}, \ \forall b \in N, \ \forall j \in N.$$

Proof. From lemma 2 and from the fact that $c_k > c^2_{k-1} > c^{2^2}_{k-2} > c^{2^3}_{k-3} > \ldots > c^{2^{k-2}}_{k-(k-2)} > c_2^{2^{k-2}} > 2^{2^{k-2}}$ it follows that

$$\frac{|\Delta^{\bar{O}^2}_{a_1a_2\ldots a_k j}|}{|\Delta^{\bar{O}^2}_{a_1a_2\ldots a_k}|} < \frac{1}{c_k^2} < \frac{1}{(2^{2^{k-2}})^2} = \frac{1}{2^{2^{k-1}}}.$$

Similarly, taking into account the fact that $c_k > c_2^{k-2}$ and $c_2 \geq b(b+1) > b^2$, we get

$$\frac{|\Delta^{\bar{O}^2}_{ba_2\ldots a_k j}|}{|\Delta^{\bar{O}^2}_{ba_2\ldots a_k}|} < \frac{1}{c_k^2} < \frac{1}{c_2^{k-1}} < \frac{1}{(b^{2^{k-1}})^2} = \frac{1}{b^{2^k}}, \ \forall b \in N.$$

□

3. Properties of symbolic dynamics related to the $\bar{O}^2$-expansion

Let us consider the dynamical system generated by the one-sided shift transformation T on the $\bar{O}^2$-expansion:

$$\forall \ x = \bar{O}^2(d_1(x), d_2(x), \ldots, d_n(x), \ldots) \quad \in [0,1],$$

$$T(x) = \bar{O}^2(d_2(x), d_3(x), \ldots, d_n(x), \ldots).$$

Recall that a set A is said to be invariant w.r.t. a measurable transformation T, if $A = T^{-1}A$. A measure μ is said to be ergodic w.r.t. a transformation T, if any invariant set $A \in \mathfrak{B}$ is either of full or of zero measure μ. A measure μ is said to be invariant w.r.t. a transformation T, if for any set $E \in \mathfrak{B}$ one has $\mu(T^{-1}E) = \mu(E)$.

It is well known that the development of metric and ergodic theories of some expansion for reals can be essentially simplified if one can find a measure which is invariant and ergodic w.r.t. one-sided shift transformation on the corresponding expansion and absolutely continuous w.r.t. Lebesgue measure. For instance, having the Gauss measure (i.e., the measure with the density $f(x) = \frac{1}{\ln 2}\frac{1}{1+x}$ on the unit interval) as invariant and ergodic measure w.r.t. the transformation $T(x) = \frac{1}{x}(mod 1)$, one can easily derive main metric and ergodic properties of the continued fraction (c.f.)-expansion. In this section we shall try to find the $\bar{O}^2$-analogue of the Gauss measure.

Let $N_i(x,k)$ be the number of the symbols "i" among the first k symbols of the $\bar{O}^2$-expansion of x. If the limit $\lim\limits_{k\to\infty} \frac{N_i(x,k)}{k}$ exists, then it is said to be the asymptotic frequency of the digit (symbol) "i" in the $\bar{O}^2$-expansion of the real number x. We shall use the notation $\nu_i(x, \bar{O}^2)$, or $\nu_i(x)$ for cases where no confusion can arise.

Theorem 1. *In the $\bar{O}^2$- expansion of almost all real numbers from the unit interval any digit i from the alphabet $A = \mathbb{N}$ appears only finitely many times, i.e., for Lebesgue almost all real numbers $x \in [0,1]$ and for any symbol $i \in \mathbb{N}$ one has:*

$$\limsup_{n\to\infty} N_i(x,n) < +\infty.$$

Proof. Let $N_i(x)$ be the number of symbols "i" in the $\bar{O}^2$-expansion of the real number x. We shall prove that the Lebesgue measure of the set

$$A_i = \{x : \ N_i(x) = \infty\}$$

is equal to zero for any $i \in N$.

To this end let us consider the set

$$\bar{\Delta}_i^k = \{x : x = \bar{O}^2(d_1, \ldots, d_{k-1}, i, d_{k+1}, \ldots); d_j \in N, \forall j \neq k\},$$

of all real numbers whose $\bar{O}^2$-expansion contains the digit i at the k-th position, i.e., $d_k(x) = i$.

Lemma 4. *For any $i \in N$ and $k \in N$ one has*

$$\lambda\left(\bar{\Delta}_i^1\right) = \frac{1}{i(i+1)} \leq \frac{1}{2},$$

$$\lambda(\bar{\Delta}_i^k) \leq \frac{1}{2^{2^{k-2}}} \ \textit{for } k > 1.$$

Proof. Since $\bar{\Delta}_i^1 = \Delta_i^{\bar{O}^2} = \left[\frac{1}{i+1}; \frac{1}{i}\right]$, we have $\lambda\left(\bar{\Delta}_i^1\right) = \frac{1}{i(i+1)} \leq \frac{1}{2}$.

From the above mentioned properties of cylindrical sets and from the definition of the set $\bar{\Delta}_i^k$ it follows that

$$\bar{\Delta}_i^k = \bigcup_{a_1=1}^{\infty} \cdots \bigcup_{a_{k-1}=1}^{\infty} \Delta_{a_1\ldots a_{k-1}i}^{\bar{O}^2}$$

and

$$\frac{|\Delta_{a_1\ldots a_{k-1}i}^{\bar{O}^2}|}{|\Delta_{a_1\ldots a_{k-1}}^{\bar{O}^2}|} \leq \frac{1}{2^{2^{k-2}}},$$

Therefore

$$\lambda\left(\bar{\Delta}_i^k\right) = \sum_{a_1=1}^{\infty} \cdots \sum_{a_{k-1}=1}^{\infty} |\Delta_{a_1\ldots a_{k-1}i}^{\bar{O}^2}| \leq \frac{1}{2^{2^{k-2}}} \left(\sum_{a_1=1}^{\infty} \cdots \sum_{a_{k-1}=1}^{\infty} |\Delta_{a_1\ldots a_{k-1}}^{\bar{O}^2}|\right) = \frac{1}{2^{2^{k-2}}}.$$

□

It is clear that A_i is the upper limit of the sequence of the set $\{\bar{\Delta}_i^k\}$, i.e.,

$$A_i = \limsup_{k\to\infty} \bar{\Delta}_i^k = \bigcap_{m=1}^{\infty} \left(\bigcup_{k=m}^{\infty} \bar{\Delta}_i^k\right).$$

Taking into account the Borel-Cantelli Lemma and the fact that

$$\sum_{k=1}^{\infty} \lambda(\bar{\Delta}_i^k) \leq \sum_{k=1}^{\infty} \frac{1}{2^{2^{k-2}}} < +\infty,$$

we deduce that

$$\lambda(A_i) = 0, \forall i \in N.$$

So,

$$\lambda(\bar{A}_i) = 1, \forall i \in N.$$

Let

$$\bar{A} = \bigcap_{i=1}^{\infty} \bar{A}_i.$$

Then $\lambda(\bar{A}) = 1$, which proves the theorem. □

Corollary 3. *For Lebesgue almost all real numbers x from the unit interval and $\forall i \in N : \nu_i(x) = 0$.*

Corollary 4. *For any stochastic vector $\overrightarrow{p} = (p_1, p_2, \ldots, p_k, \ldots)$ the set*

$$I_{\overrightarrow{p}} = \{x : \ x = \bar{O}^2(d_1(x), \ldots, d_k(x), \ldots), \ \nu_i(x) = p_i \ \ \forall i \in N\}$$

is of zero Lebesgue measure.

Theorem 2. *There are no probability measures which are simultaneously invariant and ergodic w.r.t the one-sided shift transformation T on the $\bar{O}^2$-expansion, and absolutely continuous w.r.t. Lebesgue measure.*

Proof. To prove the theorem ad absurdum, let us assume that there exists an absolutely continuous probability measure ν, which is invariant and ergodic w.r.t. the above defined transformation T. Then for ν-almost all $x \in [0, 1]$ (and, so, for a set of positive Lebesgue measure) and for any function $\varphi \in L^1([0, 1], \nu)$ the following equality holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j(x)) = \int_0^1 \varphi(x) d(\nu(x)) = \int_0^1 \varphi(x) f_\nu(x) dx,$$

where $f_\nu(x)$ is the density of the measure ν.

Let $\varphi_i(x) = 1$, if $x \in \Delta_i^{\bar{O}^2}$; and $\varphi_i(x) = 0$ otherwise.

Then the condition

$$\int_0^1 \varphi_i(x) f_\nu(x) dx = \int_{x\in\Delta_i^{\bar{O}^2}} f_\nu(x) dx > 0 \text{ holds at least for one digit } i \in N.$$

Let the latter condition hold for some fixed number i_0.

On the other hand we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_{i_0}(T^j(x)) = \lim_{n\to\infty} \frac{N_{i_0}(x, n)}{n} = 0$$

for λ-almost all $x \in [0, 1]$.

So,

$$\lim_{n\to\infty}\frac{N_{i_0}(x,n)}{n}=0$$

for λ-almost all $x\in[0,1]$, and simultaneously, we have

$$\lim_{n\to\infty}\frac{N_{i_0}(x,n)}{n}>0$$

for a set of positive Lebesgue measure. This contradiction proves the theorem. □

Corollary 5. *The application of the classical "king approach" to the development of the metric theory (i.e., the exploitation of the Birkgoff ergodic theorem with a measure which is absolutely continuous w.r.t. Lebesgue measure and which is ergodic and invariant w.r.t. the one-sided shift transformation on the corresponding expansion) is impossible for the case of the $\bar{O}^2$-expansion for real numbers.*

4. On singularity of the random $\bar{O}^2$-expansion

Let

$$\eta=\bar{O}^2(\eta_1,\eta_2,\dots,\eta_k,\dots),\tag{4}$$

be the random $\bar{O}^2$-expansion with independent identically distributed $\bar{O}^2$-symbols η_k, i.e. η_k are i.i.d. random variables taking values 1, 2, …, m, … with probabilities p_1, p_2, …, p_m, … correspondingly, with $p_m\geq 0$, $\sum\limits_{m=1}^{\infty}p_m=1$.

The following theorem completely describes the Lebesgue decomposition of the distribution of η.

Theorem 3. *The random variable η with independent identically distributed increments of the second Ostrogradsy expansion (i.e., the random variable defined by equality (4) is:*

1) of pure atomic distribution with a unique atom (iff $p_i=1$ for some $i\in N$);

2) or it is singularly continuously distributed (in all other cases).

Proof. 1) The first statement of the theorem is evident and we added it only for the completeness of the Lebesgue classification.

2) To prove the second statement we firstly remark that the probability measure μ_η is invariant and ergodic w.r.t. T. So, from the latter Theorem it follows that μ_η can not be absolutely continuous. Let us show now that in the case of continuity the distribution of η can not contain an absolutely continuous component.

Secondly let us remark that a direct application of the strong law of large numbers shows that for μ_η - almost all $x\in[0,1]$ the following equality holds

$$\nu_i(x,\bar{O}^2)=p_i,\quad\forall i\in N.\tag{5}$$

Let us choose a positive integer i_0 such that $p_{i_0} > 0$ (there exists at least one such a number) and let us consider the set $M_{i_0} = \{x : x \in [0,1], \nu_{i_0}(x, \bar{O}^2) = p_{i_0} > 0\}$. From (5) it follows that this set is of full μ_η-measure.

Let us also consider the set $L^*_{i_0} = \{x : x \in [0,1],\ \nu_{i_0}(x, \bar{O}^2) = 0\}$. From the corollary after Theorem 1 it follows that $\lambda(L^*_{i_0}) = 1$. The sets M_{i_0} and $L^*_{i_0}$ do not intersect. The first of them is a support of the probability measure μ_η, and the second one is the support of the Lebesgue measure on the unit interval. So, the measure μ_η is purely singular w.r.t. Lebesgue measure, which proves the theorem. □

5. Cantor-like sets related to the $\bar{O}^2$-expansion

Let $V_k \subset N$ and let

$$C = C[\bar{O}^2, \{V_k\}] = \{x : \ x = \bar{O}^2(d_1, d_2, \ldots, d_k, \ldots), d_k \in V_k, \forall k \in N\}.$$

The main goal of this section is to study metric and fractal properties of the set $C[\bar{O}^2, \{V_k\}]$ and their dependence on the sequence of sets $\{V_k\}$.

It is clear that the set $C[\bar{O}^2, \{V_k\}]$ is nowhere dense if and only if the condition $V_k \neq N$ holds for infinitely many indices k.

Let

$$F_k = \{x : \ x \in \Delta^{\bar{O}^2}_{a_1 a_2 \ldots a_k},\ a_j \in V_j, j = \overline{1,k}\},$$

$$\overline{F}_{k+1} = \{x : \ x \in \Delta^{\bar{O}^2}_{a_1 a_2 \ldots a_k a_{k+1}},\ a_j \in V_j, j = \overline{1,k}, a_{k+1} \notin V_{k+1}\}$$

$$\lambda(\overline{F}_{k+1}) = \lambda(F_k) - \lambda(F_{k+1})$$

$$\frac{\lambda(\overline{F}_{k+1})}{\lambda(F_k)} = 1 - \frac{\lambda(F_{k+1})}{\lambda(F_k)}$$

Lemma 5.

$$\lambda(C) = \prod_{k=1}^{\infty} \left[1 - \frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}\right].$$

Proof.

$$\lambda(C) = \lim_{k\to\infty} \lambda(F_k) = \lim_{k\to\infty} \frac{\lambda(F_k)}{\lambda(F_{k-1})} \cdot \frac{\lambda(F_{k-1})}{\lambda(F_{k-2})} \cdot \ldots \cdot \frac{\lambda(F_2)}{\lambda(F_1)} \cdot \frac{\lambda(F_1)}{\lambda(F_0)} =$$

$$= \lim_{k\to\infty} \prod_{i=1}^{k} \frac{\lambda(F_i)}{\lambda(F_{i-1})} = \prod_{k=1}^{\infty} \frac{\lambda(F_k)}{\lambda(F_{k-1})} = \prod_{k=1}^{\infty} \left[1 - \frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}\right].$$

□

Corollary 6.

$$\lambda\left(C[\bar{O}^2, \{V_k\}]\right) > 0 \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} \frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})} < +\infty;$$

$$\lambda\left(C[\bar{O}^2,\{V_k\}]\right)=0 \quad \Leftrightarrow \quad \sum_{k=1}^{\infty}\frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}=\infty.$$

a) First of all let us consider the case

$$V_k=\{v_k+1,v_k+2,\ldots\},$$

where $\{v_k\}$ is an arbitrary sequence of positive integers.

Theorem 4. *Let $V_k=\{v_k+l, l\in N\}$, where $v_k\in N$. If there exists a positive integer b such that*

$$\sum_{k=1}^{\infty}\frac{v_{k+1}}{b^{2^k}}<+\infty,$$

then $\lambda(C[\bar{O}^2,\{V_k\}])>0$.

Proof. Let c be an arbitrary positive integer such that $c>max\{v_1,b\}$ and let us consider the cylinder $\Delta_c^{\bar{O}^2}$ of the first rank. Then

$$\lambda\left(\overline{F}_{k+1}\bigcap\Delta_c^{\bar{O}^2}\right)=\sum_{a_2\in\ V_2,\ \ldots,\ a_k\in\ V_k}\ \sum_{j=1}^{v_{k+1}}|\overline{\Delta}_{ca_2\ldots a_kj}|.$$

Taking into account Lemma 3, we have

$$\sum_{j=1}^{v_{k+1}}|\overline{\Delta}_{ca_2\ldots a_kj}|\le v_{k+1}\cdot|\overline{\Delta}_{ca_2\ldots a_k1}|<v_{k+1}\cdot\frac{1}{b^{2^k}}\cdot|\overline{\Delta}_{ca_2\ldots a_k}|,$$

and therefore

$$\frac{\lambda\left(\overline{F}_{k+1}\bigcap\Delta_c^{\bar{O}^2}\right)}{\lambda\left(F_k\bigcap\Delta_c^{\bar{O}^2}\right)}=\frac{\sum\limits_{a_2\in\ V_2,\ \ldots,\ a_k\in\ V_k}\ \sum\limits_{j=1}^{v_{k+1}}|\overline{\Delta}_{ca_2\ldots a_kj}|}{\sum\limits_{a_2\in\ V_2,\ \ldots,\ a_k\in\ V_k}|\overline{\Delta}_{ca_2\ldots a_k}|}\le$$

$$\le\frac{\sum\limits_{a_2\in\ V_2,\ \ldots,\ a_k\in\ V_k}\frac{v_{k+1}}{b^{2^k}}|\overline{\Delta}_{ca_2\ldots a_k}|}{\sum\limits_{a_2\in\ V_2,\ \ldots,\ a_k\in\ V_k}|\overline{\Delta}_{ca_2\ldots a_k}|}=\frac{v_{k+1}}{b^{2^k}}.$$

From the construction of the set $C[\bar{O}^2,\{V_k\}]$ it follows that

$$\lambda(C[\bar{O}^2,\{V_k\}])\ge\lambda(C[\bar{O}^2,\{V_k\}]\bigcap\Delta_c^{\bar{O}^2})=\lim_{k\to\infty}\lambda(F_k\bigcap\Delta_c^{\bar{O}^2})=$$

$$=\lim_{k\to\infty}\frac{\lambda(F_k\bigcap\Delta_c^{\bar{O}^2})}{\lambda(F_{k-1}\bigcap\Delta_c^{\bar{O}^2})}\cdot\frac{\lambda(F_{k-1}\bigcap\Delta_c^{\bar{O}^2})}{\lambda(F_{k-2}\bigcap\Delta_c^{\bar{O}^2})}\cdot\ldots\cdot\frac{\lambda(F_2\bigcap\Delta_c^{\bar{O}^2})}{\lambda(F_1\bigcap\Delta_c^{\bar{O}^2})}\cdot\frac{\lambda(F_1\bigcap\Delta_c^{\bar{O}^2})}{\lambda(F_0\bigcap\Delta_c^{\bar{O}^2})}=$$

$$=\prod_{k=1}^{\infty}\frac{\lambda(F_k\bigcap\Delta_c^{\bar{O}^2})}{\lambda(F_{k-1}\bigcap\Delta_c^{\bar{O}^2})}=\prod_{k=1}^{\infty}\left[1-\frac{\lambda(\overline{F}_k\bigcap\Delta_c^{\bar{O}^2})}{\lambda(F_{k-1}\bigcap\Delta_c^{\bar{O}^2})}\right].$$

Since

$$\sum_{k=1}^{\infty}\frac{\lambda\left(\overline{F}_{k+1}\bigcap\Delta_c^{\bar{O}^2}\right)}{\lambda\left(F_k\bigcap\Delta_c^{\bar{O}^2}\right)}\le\sum_{k=1}^{\infty}\frac{v_{k+1}}{b^{2^k}}$$

and the series $\sum\limits_{k=1}^{\infty}\frac{v_{k+1}}{b^{2^k}}$ converges, we have $\lambda\left(C[\bar{O}^2,\{V_k\}]\right)\geq\lambda\left(C[\bar{O}^2,\{V_k\}]\bigcap\Delta_c^{\bar{O}^2}\right)>0$, which proves the Theorem. □

Proposition 1. *If $V_k=\{m+1,m+2,m+3,...\}$ for some $m\in N$, then*

$$\lambda(C[\bar{O}^1,\{V_k\}])>0; \tag{6}$$

$$\lambda(C[c.f.,\{V_k\}])=0; \tag{7}$$

$$\lambda(C[\bar{O}^2,\{V_k\}])>0. \tag{8}$$

Proof. Inequality (6) follows from Theorem 5 of the paper [1].

To prove (7) let us remind that for λ-almost all $x\in[0,1]$ a given digit "i" appears in the continued fraction expansion of the real number x with the asymptotic frequency $\frac{1}{\ln 2}\cdot\ln\frac{(i+1)^2}{i(i+2)}$. This fact is actually a direct corollary from the Birkgoff ergodic theorem and the fact that the Gauss measure $G(E):=\frac{1}{\ln 2}\int\limits_E\frac{1}{1+x}dx$ is invariant and ergodic w.r.t. one-sided shift transformation on the continued fraction expansion.

If $i=1$, then

$$\nu_1^{c.f.}(x)=\frac{1}{\ln 2}\ln\frac{4}{3} \tag{9}$$

for λ-almost all $x\in[0,1]$. On the other hand,

$$\nu_1^{c.f.}(x)=0,\ \forall x\in C[c.f.,\{V_k\}],$$

which proves equality (7).

Finally, inequality (8) is a direct corollary of the latter theorem. □

Proposition 2. *If $V_k=\{3^k+1,3^k+2,3^k+3,...\}$, $k\in N$, then*

$$\lambda(C[\bar{O}^1,\{V_k\}])=0; \tag{10}$$

$$\lambda(C[c.f.,\{V_k\}])=0; \tag{11}$$

$$\lambda(C[\bar{O}^2,\{V_k\}])>0. \tag{12}$$

Proof. Equality (11) follows from the proof of the previous proposition and inequality (12) is a direct consequence of the latter theorem.

To prove (10), let us prove two auxiliary lemmas, which have a self standing interest.

Lemma 6. *Let $\bar{O}^1_{g_1(x)g_2(x)...g_n(x)...}$ be the Ostrogradskyi-Pierce expansion in the difference form, $g_n(x)\in N$.*

Then for λ-almost all $x\in[0,1]$ one has

$$\overline{\lim_{n\to\infty}}\sqrt[n]{g_n(x)}\leq e \tag{13}$$

Proof. In [27] it has been proven that for the standard Ostrogradskyi-Pierce expansion $\bar{O}_{q_1(x)q_2(x)...q_n(x)...}$ ($q_{n+1}>q_n$) for almost all (in the sense of Lebesgue measure) $x\in[0,1]$ one has:

$$\lim_{n\to\infty}\sqrt[n]{q_n(x)}=e, \tag{14}$$

where $q_n(x) = g_1(x) + g_2(x) + \ldots + g_n(x)$.

From (14) it follows that $\forall\varepsilon > 0 \ \exists \ N(\varepsilon, x) > 0 : \quad \forall n > N(\varepsilon, x)$

$$(e-\varepsilon)^n < q_n(x) < (e+\varepsilon)^n$$

$$(e-\varepsilon)^{n+1} < q_{n+1}(x) < (e+\varepsilon)^{n+1}$$

So, $g_n(x) = q_{n+1}(x) - q_n(x) < (e+\varepsilon)^{n+1} - (e-\varepsilon)^n < (e+\varepsilon)^{n+1}$. Therefore,

$$\sqrt[n]{g_n(x)} < (e+\varepsilon)^{1+\frac{1}{n}}, \ \forall n > N(\varepsilon, x),$$

and, hence,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{g_n(x)} \leq e,$$

which proves the lemma. $\square$

Lemma 7. *If*

$$V_k = \{v_k + l, l \in N\} \tag{15}$$

and

$$\underline{\lim}_{k\to\infty} \sqrt[k]{v_k} > e, \tag{16}$$

then

$$\lambda(C[\bar{O}^1, \{V_k\}]) = 0.$$

Proof. From (15) it follows that $g_k(x) > v_k, \ \forall x \in C[\bar{O}^1, \{V_k\}]$. Therefore,

$$\overline{\lim_{k\to\infty}} \sqrt[k]{g_k(x)} \geq \underline{\lim}_{k\to\infty} \sqrt[k]{v_k} > e, \ \forall x \in C[\bar{O}^1, \{V_k\}].$$

Taking into account the latter lemma, we get

$$\lambda\left(C[\bar{O}^1, \{V_k\}]\right) = 0.$$

$\square$

So, to show inequality (10), it is enough to apply lemma 7 with $v_k = 3^k$. $\square$

Remark. Proposition 1 shows essential differences between the metric theory of continued fractions and the metric theory of the $\bar{O}^2$-expansion. At the same time this proposition shows some level of similarity between $\bar{O}^1$- and $\bar{O}^2$-expansions of real numbers. Indeed, from (6) and (8) it follows that the deleting of any finite number of digits from the alphabet does not affect the positivity of the Lebesgue measure of the set $C[\bar{O}^1, \{V_k\}]$, and therefore, does not change the Hausdorff dimension of this set. Moreover, both the mapping $f_1 : E \to E\bigcap C[\bar{O}^1, \{1, 2, ..., m\}]$ and the mapping $f_2 : E \to E\bigcap C[\bar{O}^2, \{1, 2, ..., m\}]$ do not change the Hausdorff dimension of any subset $E \subset [0, 1]$.

On the other hand Proposition 2 demonstrates obvious differences in the metric theories of $\bar{O}^1$- and $\bar{O}^2$-expansions. To stress these differences, let us mention that the transformations $T_{\bar{O}^2\to\bar{O}^1}$ and $T_{\bar{O}^2\to c.f.}$

are strictly increasing **singularly** continuous functions on the unit interval, where

$$T_{\bar{O}^2\to\bar{O}^1}(\Delta^{\bar{O}^2}_{g_1(x)g_2(x)...g_k(x)...}) = \Delta^{\bar{O}^1}_{g_1(x)g_2(x)...g_k(x)...}, \tag{17}$$

and

$$T_{\bar{O}^2\to c.f.}(\Delta^{\bar{O}^2}_{g_1(x)g_2(x)...g_k(x)...}) = \Delta^{c.f.}_{g_1(x)g_2(x)...g_k(x)...}. \tag{18}$$

Actually the singularity of the function $T_{\bar{O}^2\to\bar{O}^1}$ follows from (10) and (12). The singularity of the function $T_{\bar{O}^2\to c.f.}$ follows from (9) and from the fact that for λ-almost all $x \in [0,1]$ the asymptotic frequency of the digit 1 in the $\bar{O}^2$-expansion of x is equal to zero.

So, the continued fraction expansion, $\bar{O}^1$-expansion and $\bar{O}^2$-expansion are "mutually orthogonal", and their metric theories differ essentially.

b) Now let us consider the case where

$$V_k = \{1, 2, \dots, m_k\} \tag{19}$$

and $\{m_k\}$ is an arbitrary sequence of positive integers.

Let $M_1 := 1$, and

$$M_{k+1} = (M_k + 1)^2 + m_{k+1},\ \forall k \in N.$$

Theorem 5. *If* $\sum\limits_{k=1}^{\infty} \frac{M_k^2}{m_{k+1}} < +\infty$, *then* $\lambda\left(C[\bar{O}^2, \{V_k\}]\right) > 0$.

Proof. Let $\Delta^{\bar{O}^2}_{a_1...a_k}$ be an arbitrary cylinder of rank k with $a_j \in V_j, \forall j \in \{1, 2, ..., k\}$. Then

$$\lambda\left(\overline{F}_{k+1}\bigcap\Delta^{\bar{O}^2}_{a_1...a_k}\right) = \sum_{i=m_{k+1}+1}^{\infty} |\Delta^{\bar{O}^2}_{a_1...a_k i}| = \frac{1}{c_k(c_k+1)+m_{k+1}},$$

where c_k can be calculated via $a_1, a_2, \dots, a_k$: $c_{i+1} = c_i(c_i+1)-1+a_{i+1}$ with $c_1 = a_1$. It is clear that

$$\lambda\left(F_k\bigcap\Delta^{\bar{O}^2}_{a_1...a_k}\right) = \lambda\left(\Delta^{\bar{O}^2}_{a_1...a_k}\right) = \frac{1}{c_k(c_k+1)},\ a_j \in V_j, j = \overline{1,k}.$$

Therefore

$$\frac{\lambda\left(\overline{F}_{k+1}\bigcap\Delta^{\bar{O}^2}_{a_1...a_k}\right)}{\lambda\left(F_k\bigcap\Delta^{\bar{O}^2}_{a_1...a_k}\right)} = \frac{\frac{1}{c_k(c_k+1)+m_{k+1}}}{\frac{1}{c_k(c_k+1)}} = \frac{c_k(c_k+1)}{c_k(c_k+1)+m_{k+1}}.$$

From (19) it follows that

$$1 \le c_1 \le m_1,$$

$$2 \le c_2 \le c_1(c_1+1)-1+m_2 \le (m_1+1)^2+m_2,$$

$$2\cdot 3 \le c_3 \le c_2(c_2+1)-1+m_3 \le (c_2+1)^2+m_3 = \left((m_1+1)^2+m_2\right)^2+m_3,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$2^{2^{k-2}} \le c_k \le (((m_1+1)^2+m_2)^2+\dots+m_{k-1})^2+m_k, ...$$

So,

$$2^{2^{k-2}} \le c_k \le M_k, \forall k \in N.$$

Therefore

$$\frac{2^{2^{k-2}}(2^{2^{k-2}}+1)}{2^{2^{k-2}}(2^{2^{k-2}}+1)+m_{k+1}} \leq \frac{\lambda\left(\overline{F}_{k+1}\bigcap\Delta^{\bar{O}^2}_{a_1\ldots a_k}\right)}{\lambda\left(F_k\bigcap\Delta^{\bar{O}^2}_{a_1\ldots a_k}\right)} \leq \frac{M_k(M_k+1)}{M_k(M_k+1)+m_{k+1}}. \tag{20}$$

Since the estimation (20) holds for any cylinder $\Delta^{\bar{O}^2}_{a_1\ldots a_k}$, $a_j \in V_j, j=\overline{1,k}$, we get

$$\frac{2^{2^{k-1}}}{2^{2^{k-1}}+m_{k+1}} \leq \frac{\lambda(\overline{F}_{k+1})}{\lambda(F_k)} \leq \frac{(M_k+1)^2}{(M_k+1)^2+m_{k+1}}.$$

If $\sum\limits_{k=1}^{\infty}\frac{(M_k+1)^2}{(M_k+1)^2+m_{k+1}} < +\infty$, then $\sum\limits_{k=1}^{\infty}\frac{\lambda(\overline{F}_{k+1})}{\lambda(F_k)} < +\infty$, and taking into account corollary 6 we deduce that $\lambda\left(C[\bar{O}^2, V_k]\right) > 0$. One can easily verify that

$$\frac{1}{1+\frac{m_{k+1}}{M_k^2}} = \frac{M_k^2}{M_k^2+m_{k+1}} \leq \frac{(M_k+1)^2}{(M_k+1)^2+m_{k+1}} = \frac{\left(\frac{M_k+1}{M_k}\right)^2\cdot M_k^2}{\left(\frac{M_k+1}{M_k}\right)^2\cdot M_k^2+m_{k+1}} \leq$$

$$\leq \frac{4M_k^2}{4M_k^2+m_{k+1}} \leq \frac{1}{1+\frac{m_{k+1}}{4M_k^2}} \leq \frac{4M_k^2}{m_{k+1}}.$$

So,

$$\sum_{k=1}^{\infty}\frac{(M_k+1)^2}{(M_k+1)^2+m_{k+1}} < +\infty \ \Leftrightarrow\ \sum_{k=1}^{\infty}\frac{M_k^2}{m_{k+1}} < +\infty.$$

Therefore the convergence of the series $\sum\limits_{k=1}^{\infty}\frac{M_k^2}{m_{k+1}}$ implies the positivity of the Lebesgue measure of the set $C[\bar{O}^2, \{V_k\}]$. □

Theorem 6. *If* $\sum\limits_{k=1}^{\infty}\frac{2^{2^{k-1}}}{2^{2^{k-1}}+m_{k+1}} = +\infty$, *then* $\lambda\left(C[\bar{O}^2, \{V_k\}]\right) = 0$.

Proof. Using the estimation (20) we have

$$\frac{\lambda\left(\overline{F}_{k+1}\bigcap\Delta^{\bar{O}^2}_{a_1\ldots a_k}\right)}{\lambda\left(F_k\bigcap\Delta^{\bar{O}^2}_{a_1\ldots a_k}\right)} \geq \frac{2^{2^{k-1}}}{2^{2^{k-1}}+m_{k+1}}$$

for all cylinders $\Delta^{\bar{O}^2}_{a_1\ldots a_k}$, $a_j \in V_j, j=\overline{1,k}$.

Therefore

$$\frac{\lambda\left(\overline{F}_{k+1}\right)}{\lambda(F_k)} \geq \frac{2^{2^{k-1}}}{2^{2^{k-1}}+m_{k+1}}, \ \forall k \in N.$$

If $\sum\limits_{k=1}^{\infty}\frac{2^{2^{k-1}}}{2^{2^{k-1}}+m_{k+1}} = +\infty$, then, applying corollary 6, we get $\lambda\left(C[\bar{O}^2, \{V_k\}]\right) = 0$, which proves the Theorem □

Proposition 3. *Let $V_k = \{1, 2, ..., 2^{2^{k-1}}\}$. Then*

$$\lambda(C[\bar{O}^1, \{V_k\}]) > 0; \tag{21}$$

$$\lambda(C[c.f., \{V_k\}]) > 0; \tag{22}$$

$$\lambda(C[\bar{O}^2, \{V_k\}]) = 0. \tag{23}$$

Proof. Equality (23) is a direct corollary of the latter theorem.

To prove inequality (21), let us remind (see, e.g., [14]) that the condition $\sum\limits_{k=1}^{\infty} \frac{m_1+m_2+...+m_k}{m_{k+1}} < +\infty$ implies the positivity of the Lebesgue measure of the set $C[\bar{O}^1, \{V_k\}]$ with $V_k = \{1, 2, ..., m_k\}$. Since the series $\sum\limits_{k=1}^{\infty} \frac{m_1+m_2+...+m_k}{m_{k+1}}$ diverges for $m_k = 2^{2^{k-1}}$, we get (21).

To prove inequality (22), let us remind (see, e.g., [22]) that

$$\frac{1}{3i^2} \leq \frac{|\Delta^{c.f.}_{a_1...a_n i}|}{|\Delta^{c.f.}_{a_1...a_n}|} \leq \frac{1}{i^2}.$$

So,

$$\frac{\sum\limits_{i \notin V_{k+1}} |\Delta^{c.f.}_{a_1...a_n i}|}{|\Delta^{c.f.}_{a_1...a_n}|} \leq \sum_{i=2^{2^{k-1}}+1}^{\infty} \frac{2}{i^2} < \frac{4}{2^{2^{k-2}}},$$

and therefore

$$\sum_{k=1}^{\infty} \frac{\lambda(\overline{F}^{c.f.}_{k+1})}{\lambda(F^{c.f.}_k)} \leq \sum_{k=1}^{\infty} \frac{4}{2^{2^{k-2}}} < +\infty,$$

which implies the positivity of the Lebesgue measure of the set $\lambda(C[c.f., \{V_k\}])$. □

c) Finally let us consider the case where both the set V_k and the set $\overline{V}_k := N \setminus V_k$ are infinite for any $k_i n N$.

Theorem 7. *Let $V_k = N \setminus \{b_1^{(k)}, b_2^{(k)}, \ldots, b_m^{(k)}, \ldots\}$, where $\{b_m^{(k)}\}_{m=1}^{\infty}$ is an increasing sequence of positive integers $\forall k \in N$, and let for any $k \in N$ exist a positive integer $d_k \in N$ such that*

$$b^k_{n+1} - b^k_n \leq d_k, \ \forall n \in N. \tag{24}$$

Then if

$$\sum_{k=1}^{\infty} \frac{1}{b_1^{(k)} \cdot d_k} = +\infty, \tag{25}$$

then $\lambda\left(C[\bar{O}^2, \{V_k\}]\right) = 0$.

Proof. Let $\Delta^{\bar{O}_2}_{a_1 a_2 \ldots a_{k-1}}$ be an arbitrary cylinder of rank $k-1$ with $a_j \in V_j, \ \forall j \in \{1, 2, \ldots, k-1\}$. Then

$$|\Delta^{\bar{O}^2}_{a_1 a_2 \ldots a_{k-1} b_1^{(k)}}| \geq |\Delta^{\bar{O}^2}_{a_1 a_2 \ldots a_{k-1} (b_1^{(k)}+1)}| \geq \ldots \geq |\Delta^{\bar{O}^2}_{a_1 a_2 \ldots a_{k-1} (b_2^{(k)}-1)}|.$$

Therefore

$$|\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}b_1^{(k)}}| \geq \frac{1}{b_2^{(k)}-b_1^{(k)}-1}\cdot \sum_{i=b_1^{(k)}+1}^{b_2^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}|$$

On the other hand,

$$\sum_{i=1}^{b_1^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}| = \sum_{i=1}^{b_1^{(k)}-1} \frac{1}{(c_{k-1}(c_{k-1}+1)-1+i)(c_{k-1}(c_{k-1}+1)+i)} =$$

$$= \sum_{i=1}^{b_1^{(k)}-1}\left(\frac{1}{c_{k-1}(c_{k-1}+1)-1+i}-\frac{1}{c_{k-1}(c_{k-1}+1)+i}\right) =$$

$$= \frac{1}{c_{k-1}(c_{k-1}+1)} - \frac{1}{c_{k-1}(c_{k-1}+1)+b_1^{(k)}-1} = \frac{b_1^{(k)}-1}{c_{k-1}(c_{k-1}+1)(c_{k-1}(c_{k-1}+1)+b_1^{(k)}-1)}.$$

Hence

$$\frac{\sum_{i=1}^{b_1^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}|}{|\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}(b_1^{(k)})}|} = \frac{\frac{b_1^{(k)}-1}{c_{k-1}(c_{k-1}+1)\left(c_{k-1}(c_{k-1}+1)+b_1^{(k)}-1\right)}}{\frac{1}{\left(c_{k-1}(c_{k-1}+1)+b_1^{(k)}-1\right)\left(c_{k-1}(c_{k-1}+1)+b_1^{(k)}\right)}} =$$

$$= \frac{c_{k-1}(c_{k-1}+1)+b_1^{(k)}}{c_{k-1}(c_{k-1}+1)}\cdot\left(b_1^{(k)}-1\right) = \left(1+\frac{b_1^{(k)}}{c_{k-1}(c_{k-1}+1)}\right)(b_1^{(k)}-1) \leq$$

$$\leq \left(1+\frac{b_1^{(k)}}{2^{2^{k-2}}}\right)\cdot b_1^{(k)} =: l_k.$$

So,

$$\begin{cases} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}b_1^{(k)}}| \geq \frac{1}{l_k}\cdot \sum\limits_{i=1}^{b_1^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}|; \\ |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}b_1^{(k)}}| \geq \frac{1}{b_2^{(k)}-b_1^{(k)}}\cdot \sum\limits_{i=b_1^{(k)}+1}^{b_2^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}|. \end{cases} \tag{26}$$

$$|\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}b_2^{(k)}}| \geq \frac{1}{b_3^{(k)}-b_2^{(k)}}\cdot \sum_{i=b_2^{(k)}+1}^{b_3^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}|;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}b_n^{(k)}}| \geq \frac{1}{b_{n+1}^{(k)}-b_n^{(k)}}\cdot \sum_{i=b_n^{(k)}+1}^{b_{n+1}^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}| \geq \frac{1}{d_k}\cdot \sum_{i=b_n^{(k)}+1}^{b_{n+1}^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}|.$$

From (26) it follows that

$$|\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}b_1^{(k)}}| \geq \frac{1}{2l_k}\cdot \sum_{i=1}^{b_1^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}| + \frac{1}{2d_k}\sum_{i=b_1^{(k)}+1}^{b_2^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}| \geq$$

$$\geq \frac{1}{2l_k \cdot d_k} \sum_{i=1,\ i\neq b_1^{(k)}}^{b_2^{(k)}-1} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}|.$$

Therefore,

$$\sum_{i\notin V_k} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}| \geq \frac{1}{2l_kd_k} \sum_{i\in V_k} |\Delta^{\bar{O}^2}_{a_1a_2\ldots a_{k-1}i}|,$$

$$\forall (a_1, a_2, \ldots, a_{k-1}),\ a_j \in V_j, j \in \{1, 2, \ldots, k-1\}.$$

Hence

$$\begin{cases} \lambda(\overline{F}_k) \geq \frac{1}{2l_kd_k} \cdot \lambda(F_k); \\ \lambda(\overline{F}_k) + \lambda(F_k) = \lambda(F_{k-1}). \end{cases}$$

So,

$$\frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})} \geq \frac{1}{2l_kd_k + 1} \geq \frac{1}{4l_kd_k}.$$

If $\sum\limits_{k=1}^{\infty} \frac{1}{l_k \cdot d_k} = +\infty$, then $\sum\limits_{k=1}^{\infty} \frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})} = +\infty$, and, therefore, $\lambda(C[\bar{O}^2, \{V_k\}]) = 0$. It is clear that

$$\sum_{k=1}^{\infty} \frac{1}{l_kd_k} = \sum_{k\in A} \frac{1}{\left(1 + \frac{b_1^{(k)}}{2^{2^{k-2}}}\right) \cdot b_1^{(k)} \cdot d_k} + \sum_{k\notin A} \frac{1}{\left(1 + \frac{b_1^{(k)}}{2^{2^{k-2}}}\right) \cdot b_1^{(k)} \cdot d_k},$$

where $A = \left\{k :\ \frac{b_1^{(k)}}{2^{2^{k-2}}} \leq 1\right\}$.

The series $\sum\limits_{k\in A} \frac{1}{\left(1+\frac{b_1^{(k)}}{2^{2^{k-2}}}\right)\cdot b_1^{(k)}\cdot d_k}$ diverges if and only if the series $\sum\limits_{k=1}^{\infty} \frac{1}{b_1^{(k)}\cdot d_k}$ does, and the series

$$\sum_{k\notin A} \frac{1}{\left(1 + \frac{b_1^{(k)}}{2^{2^{k-2}}}\right) \cdot b_1^{(k)} \cdot d_k}$$

always converges.

Therefore the divergence of the series $\sum\limits_{k=1}^{\infty} \frac{1}{l_k\cdot d_k}$ is equivalent to the divergence of the series $\sum\limits_{k=1}^{\infty} \frac{1}{b_1^{(k)}\cdot d_k}$, which proves the Theorem. □

Corollary 7. *Let* $V_k = N \setminus \{b_1, b_2, \ldots, b_m, \ldots\}$. *If*

$$\exists\ d \in N :\ b_{n+1} - b_n \leq d\ \forall n \in N,$$

then $\lambda\left(C[\bar{O}^2, \{V_k\}]\right) = 0$.

Corollary 8. *If* $V_k = N \setminus \{1, 3, 5, \ldots\}$, *then* $\lambda\left(C[\bar{O}^2, \{V_k\}]\right) = 0$.

Theorem 8. *If* $V_k = N \setminus \{1, 4, 9, \ldots, m^2, \ldots\}$, *then* $\lambda\left(C[\bar{O}^2, \{V_k\}]\right) > 0$.

Proof. Let $\Delta^{\bar{O}_2}_{a_1a_2\ldots a_{k-1}}$ be an arbitrary cylinder of rank $k-1$ with $a_j \in V_j,\ \forall j \in \{1,2,\ldots,k-1\}$. Then

$$\lambda\left(\overline{F}_k \bigcap \Delta^{\bar{O}^2}_{a_1\ldots a_{k-1}}\right) = \sum_{m=1}^{\infty} \frac{1}{(c_{k-1}(c_{k-1}+1)-1+m^2)\,(c_{k-1}(c_{k-1}+1)+m^2)},$$

$$|\Delta^{\bar{O}^2}_{a_1\ldots a_{k-1}}| = \frac{1}{c_{k-1}(c_{k-1}+1)},.$$

$$\frac{\lambda\left(\overline{F}_k \bigcap \Delta^{\bar{O}^2}_{a_1\ldots a_{k-1}}\right)}{\lambda\left(\Delta^{\bar{O}^2}_{a_1\ldots a_{k-1}}\right)} = \sum_{m=1}^{\infty} \frac{c_{k-1}(c_{k-1}+1)}{(c_{k-1}(c_{k-1}+1)-1+m^2)(c_{k-1}(c_{k-1}+1)+m^2)} <$$

$$< \sum_{m=1}^{\infty} \frac{c_{k-1}(c_{k-1}+1)-1+m^2}{(c_{k-1}(c_{k-1}+1)-1+m^2)(c_{k-1}(c_{k-1}+1)+m^2)} = \sum_{m=1}^{\infty} \frac{1}{c_{k-1}(c_{k-1}+1)+m^2} <$$

$$< \int_0^{+\infty} \frac{1}{c_{k-1}(c_{k-1}+1)+x^2}dx = \frac{\pi}{2\cdot\sqrt{(c_{k-1}(c_{k-1}+1))}} \leq \frac{\pi}{2\cdot 2^{2^{k-2}}}$$

Therefore

$$\frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})} < \frac{\pi}{2\cdot 2^{2^{k-2}}}.$$

Since

$$\sum_{k=1}^{\infty} \frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})} < +\infty,$$

we get $\lambda\left(C[\bar{O}^2, V_k]\right) > 0$,which proves the theorem. □

6. Fractal properties of real numbers with bounded $\bar{O}^2$-digits

In the case of zero Lebesgue measure, the next level for the study of properties of the sets $C[\bar{O}^2, \{V_k\}]$ is the determination of their Hausdorff dimension $\dim_H(\cdot)$ (see, e.g., [17] for the definition and main properties of this main fractal dimension).

We shall study this problem for the case where $V_k = \{1,2,...,m_k\}$. A similar problem for the continued fraction expansion were studied by many authors during last 60 years. Set

$$E_2 = \{x : x = \Delta^{c.f.}_{\alpha_1(x)\ldots\alpha_k(x)\ldots}, \alpha_k(x) \in \{1,2\}\}.$$

In 1941 Good [18] shows that

$$0,5194 < \dim_H(E_2) < 0,5433.$$

In 1982 and 1985 Bumby [15, 16] improves these bounds:

$$0,5312 < \dim_H(E_2) < 0,5314.$$

In 1989 Hensley [19] shows that

$$0,53128049 < \dim_H(E_2) < 0,53128051.$$

In 1996 the same author ([20]) improves his estimate up to

$$0,5312805062772051416.$$

A new approach to the determination of the Hausdorff dimension of the set E_2 with a desired precision was developed by Jenkinson and Policott in 2001 [21].

Our nearest aim is to study fractal properties of sets which are $\bar{O}^2$-analogues of the above discussed set E_2, i.e., the set $C[\bar{O}^2, \{1,2\}]$ and its generalization $C[\bar{O}^2, \{1,2,...,m\}]$. The following theorem shows that from the fractal geometry point of view the sets E_2 and $C[\bar{O}^2, \{1,2\}]$ (as well as their generalizations) are cardinally different.

Theorem 9. *Let $V_k = \{1,2,3,...,m_k\}$, where $m_k \in N$.*

If for any positive α the following equality

$$\lim_{k\to\infty} \frac{m_1 \cdot m_2 \cdot ... \cdot m_k}{2^{(\alpha 2^{k-1})}} = 0 \tag{27}$$

holds, then the Hausdorff dimension of the set $C[\bar{O}^2, \{V_k\}]$ is equal to zero, i.e.,

$$\dim_H(C[\bar{O}^2, \{V_k\}]) = 0.$$

Proof. From the construction of the sets V_k it follows that the set $C[\bar{O}^2, \{V_k\}]$ can be covered by m_1 cylinders of the first rank, by $m_1 \cdot m_2$ cylinders of rank 2, ..., by $m_1 \cdot m_2 \cdot \ldots \cdot m_k$ cylinders of rank k, ... The cylinder $\overline{\Delta}_{\underbrace{11\ldots1}_{k}}$ has the maximal length among all cylinders of rank k:

$$|\overline{\Delta}_{\underbrace{11\ldots1}_{k}}| \le \frac{1}{c_k(c_k+1)} < \frac{1}{2^{(2^{k-1})}}.$$

Let us consider the coverings of the set $C[\bar{O}^2, \{V_k\}]$ by cylinders of rank k:

$$\overline{\Delta}_{a_1 a_2 \ldots a_k}, \quad a_1 \in V_1, a_2 \in V_2, \ldots, a_k \in V_k.$$

It is clear that $\left(\bigcup_{a_1\in V_1,\ldots,a_k\in V_k}\right) \supset C[\bar{O}^2, \{V_k\}]$.

For a given $\varepsilon > 0$ there exists $k \in N$ such that $\frac{1}{2^{(2^{k-1})}} < \varepsilon$. In such a case the length of any k-rank cylinder of the $\bar{O}^2$-expansion does not exceed ε. So,

$$H^\alpha_\varepsilon(C[\bar{O}^2, \{V_k\}]) = \inf_{|E_i|\le\varepsilon, (\bigcup E_i)\supset C} \sum_i |E_i|^\alpha \le \sum_{a_1\in V_1,\ldots,a_k\in V_k} |\overline{\Delta}_{a_1a_2\ldots a_k}|^\alpha \le$$

$$\le m_1 \cdot m_2 \cdot \ldots \cdot m_k \cdot \left(\frac{1}{2^{(2^{k-1})}}\right)^\alpha = \frac{m_1 \cdot m_2 \cdot \ldots \cdot m_k}{2^{\alpha(2^{k-1})}}.$$

$$H^\alpha(C[\bar{O}^2, \{V_k\}]) = \lim_{\varepsilon\downarrow 0} H^\alpha_\varepsilon(C[\bar{O}^2, \{V_k\}]) = \lim_{k\to\infty} H^\alpha_{\varepsilon_k}(C[\bar{O}^2, \{V_k\}]),$$

where

$$\varepsilon_k = \frac{1}{2^{(2^{k-1})}}.$$

Since

$$H^{\alpha}_{\varepsilon_k}(C[\bar{O}^2,\{V_k\}]) \le \frac{m_1\cdot m_2\cdot\ldots\cdot m_k}{2^{\alpha(2^{k-1})}},$$

we have

$$H^{\alpha}(C[\bar{O}^2,\{V_k\}]) = \lim_{k\to\infty} H^{\alpha}_{\varepsilon_k}(C[\bar{O}^2,\{V_k\}]) \le \lim_{k\to\infty}\frac{m_1\cdot m_2\cdot\ldots\cdot m_k}{2^{\alpha(2^{k-1})}} = 0 \ (\forall\alpha>0)$$

.

Therefore, $H^{\alpha}(C[\bar{O}^2,\{V_k\}]) = 0, \quad \forall\alpha>0$, and so

$$\dim_H(C[\bar{O}^2,\{V_k\}]) = \inf\{\alpha: \quad H^{\alpha}(C[\bar{O}^2,\{V_k\}]) = 0\} = 0.$$

□

Corollary 9. *If there exists a number $a\in N$ such that $m_k \le a^k, \ \ \forall k\in N$, then*

$$\dim_H(C[\bar{O}^2,\{V_k\}]) = 0.$$

Corollary 10. *If $m_k = m_0, \ \ \forall k\in N$ for some positive integer m_0, then*

$$\dim_H(C[\bar{O}^2,\{V_k\}]) = 0.$$

Let $B(\bar{O}^2)$ be the set of all real numbers with bounded $\bar{O}^2$-symbols, i.e.,

$$B(\bar{O}^2) = \{x:\ x=\Delta_{a_1(x)\ldots a_k(x)\ldots}:\ \exists K(x)\in N:\ a_j(x)\le K(x),\ \forall j\in N\}.$$

Theorem 10. *The set $B(\bar{O}^2)$ of all numbers with bounded $\bar{O}^2$-symbols is an anomalously fractal set, i.e., the Hausdorff dimension of $B(\bar{O}^2)$ is equal to* 0*:*

$$\dim_H B(\bar{O}^2) = 0.$$

Proof. The set $B(\bar{O}^2)$ can be decomposed in the following way:

$$B(\bar{O}^2) = \bigcup_{i=1}^{\infty} B_i(\bar{O}^2),$$

where

$$B_i(\bar{O}^2) = \{x:\ a_j(x)\le i,\ \forall j\in N\}.$$

Since,

$$\dim_H B_i(\bar{O}^2) = 0,\ \forall i\in N,$$

we have

$$\dim_H B(\bar{O}^2) = \sup_i \dim_H B_i(\bar{O}^2) = 0,$$

which proves the theorem. □

Remark 2. From [23] it follows that the set of continued fractions with bounded partial quotients is of full Hausdorff dimension:

$$\dim_H(B(c.f.)) = 1,$$

which stresses the essential difference between the dimensional theories for the $\bar{O}^2$-expansion and the continued fraction expansion.

Acknowledgement This work was partly supported by DFG 436 UKR 113/97 project, EU project STREVCOMS and by the Alexander von Humboldt Foundation.

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